ON THE THEORY OF THE DIURNAL TIDE

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ABSTRACT

Laplace's tidal equation for diurnal tides of longitudinal number one is investigated. It is found that in addition to the previously found solutions (Hough Functions) corresponding to positive equivalent depths there are also Hough Functions corresponding to negative equivalent depths. Both are necessary for the representation of observed tidal data.

As an application of the Hough Functions the diurnal surface pressure oscillation resulting from diurnal variations in insolation is computed. It is found that the insolation model due to Siebert can account for only one-third of the observed pressure oscillation.

1. INTRODUCTION

In the study of atmospheric tides, the diurnal tide has been relatively neglected; the semidiurnal tide has received most of the attention. In recent years, however, with improved rocket data, it has been found that the diurnal period is dominant in mesospheric winds and temperatures (Miers [8], Beyers and Miers [2]). As a result a renewed study of the diurnal tide seemed important. Inevitably, the study of tides requires a knowledge of Hough Functions for the period and longitudinal wave number under consideration (Siebert [10]). For the diurnal tide following the sun (i.e., $\tau=+1$ day; longitudinal wave number $=+1$) the relevant Hough Functions present some interesting problems. Haurwitz [5] has recently computed three of these functions and their associated eigenvalues (i.e., equivalent depths). He found that the computation of these functions in terms of Associated Legendre Polynomials was difficult insofar as each Hough Function required a substantial number of polynomials for its accurate representation. Moreover, the amplitude of the functions he found was confined to within a region of 30° about the equator; i.e., within the critical latitudes where $f(=2\Omega \sin \theta)$ equals $2\Omega \sin 30°$ $(=2\pi/1$ day). As a result it appeared that, at best, a great number of these functions would be needed to represent globally distributed functions.

Because of the confinement to the region of 30° about the equator, it was felt by this author that an investigation of diurnal tides on an equatorially centered β-plane would yield simplified approximations to the Hough Functions and their associated equivalent depths. The resulting calculation will not be described in this paper since its results are not explicitly used. However, as was expected, good approximations to all the Hough Functions of the type found by Haurwitz and their associated equivalent depths were found. More interesting, however, was the fact that these were not the only eigensolutions. There was also an infinite set of eigenfunctions whose amplitude was concentrated outside the critical latitudes. The β-plane approximation is, of course, extremely bad at these latitudes. On the other hand, the inclusion of these eigenfunctions is necessary if the total set of functions is to be complete. This, in turn, suggests that the set of eigenfunctions used by Haurwitz [5] is not complete. A new investigation of Laplace's Tidal Equation for a spherical surface was therefore undertaken.

The remainder of this paper deals with that investigation, its results, and their application. It was found that in addition to the eigensolutions found by Haurwitz, there is, in fact, another infinite set of eigenfunctions whose amplitude is concentrated outside the critical latitudes. Associated with these eigensolutions are negative equivalent depths.[2] The existence of these eigensolutions proves the incompleteness of the original set. Moreover, the set of Hough Functions, including the new ones, proves quite suitable for the representation of observed tidal distributions.

Finally, the Hough Functions are used in order to investigate the diurnal surface pressure oscillation that should result from diurnal variations in solar insolation.

2. TIDAL EQUATIONS

We will in this section be extremely sketchy in discussing the equations, since they are developed in detail elsewhere (Wilkes [11], Siebert [10]). Briefly, the basic equations for the nth Hough mode corresponding to an oscillation of frequency ω and longitudinal wave number s are [3]

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[2] The author is indebted to A. Eliassen for showing him a recent paper by Dikii [3]. It is clear from this paper that the Russians are aware of the existence of negative equivalent depths. However, they appear to have made no use of this discovery.

[3] Oscillating gravitational potential is neglected.

$$\frac{d^2Y_n}{dx^2}-\frac{1}{4}\left[1-\frac{4}{h_n}\left(\kappa H(x)+\frac{dH}{dx}\right)\right]Y_n=\frac{\kappa J_n(x)}{gh_n}e^{-x/2} \quad (1)$$

and

$$\frac{d}{d\mu}\left(\frac{1-\mu^2}{k^2-\mu^2}\frac{dY_n}{d\mu}\right)-\frac{1}{k^2-\mu^2}\left[\frac{s}{k}\frac{k^2+\mu^2}{k^2-\mu^2}+\frac{s^2}{1-\mu^2}\right]Y_n+\frac{4a^2\Omega^2}{gh_n}Y_n=0 \quad (2)$$

where

$$Y_ne^{x/2}=\chi_n(z)-\frac{\kappa J_n(z)}{gH(z)} \quad (3)$$

$$x=\int_0^z\frac{d\zeta}{(H\zeta)} \quad (4)$$

$$\chi=\frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}(u\sin\theta)+\frac{1}{a\sin\theta}\frac{\partial v}{\partial\phi}+\frac{\partial w}{\partial z} \quad (5)$$

$$H=\frac{RT_0}{Mg}$$

$$\gamma=\frac{c_p}{c_v}=1.40$$

$$\kappa=\frac{\gamma-1}{\gamma}=\frac{2}{7}$$

R=universal gas constant,
M=molecular weight,
g=gravitational acceleration,
Ω=rotation rate of earth,
$k=\omega/2\Omega$
a=radius of earth,
J_n=nth Hough component of J,
J=heating per unit mass per unit time,
$T_0(z)$=basic temperature distribution,
z=altitude,
φ=longitude,
θ=latitude (with 0° corresponding to N. pole),
$\mu=\cos\varphi$,
u=velocity in θ-direction,
v=velocity in φ-direction,
w=velocity in z-direction,
and
h_n=equivalent depth of nth mode

The t and ϕ dependence of Y_n are given by

$$e^{i(\omega t+s\phi)}.$$

The x (or z) dependence and the μ (or θ) dependence of Y_n are separable, equation (1) giving the former and (2) the latter. In effect, h_n is the separation constant. It is common practice to use Y_n to represent only Y_n's x-dependence and Θ_n for Y_n's μ-dependence; we will adopt this practice here as well.

The conventional tidal fields may all be expressed conveniently in terms of $Y_n\Theta_n$ by the following equations:

$$u_n=\frac{\gamma gh_ne^{x/2}}{4a\Omega^2(k^2-\mu^2)}\left(\frac{dY_n(x)}{dx}-\frac{1}{2}Y_n(x)\right)\cdot\left(\frac{\partial}{\partial\theta}-\frac{i}{k}\cot\theta\frac{\partial}{\partial\phi}\right)\Theta_n(\theta)e^{i(\omega t+s\phi)} \quad (6)$$

$$v_n=\frac{i\gamma gh_ne^{x/2}}{4a\Omega^2(k^2-\mu^2)}\left(\frac{dY_n(x)}{dx}-\frac{1}{2}Y_n(x)\right)\cdot\left(\frac{\cos\theta}{k}\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)\Theta_ne^{i(\omega t+s\phi)} \quad (7)$$

$$w_n=\gamma h_ne^{x/2}\left[\frac{dY_n}{dx}+\left(\frac{H}{h_n}-\frac{1}{2}\right)Y_n\right]\Theta_ne^{i(\omega t+s\phi)} \quad (8)$$

$$\delta p_n=\frac{p_0(0)}{H}\frac{\gamma h_n}{i\omega}e^{-x/2}\left(\frac{dY_n}{dx}-\frac{1}{2}Y_n\right)\Theta_ne^{i(\omega t+s\phi)} \quad (9)$$

$$\delta T_n=\frac{M}{R}\left\{-\frac{\gamma gh_n}{i\omega}e^{x/2}\left[\frac{\kappa H}{h_n}+\frac{1}{H}\frac{dH}{dx}\left(\frac{d}{dx}+\frac{H}{h_n}-\frac{1}{2}\right)\right]Y_n+\frac{\kappa J_n}{i\omega}\right\}\Theta_ne^{i(\omega t+s\phi)}. \quad (10)$$

In order to solve equations (1) and (2) boundary conditions are needed. Boundedness over the entire latitude domain is sufficient for equation (2). This leads to an eigenfunction-eigenvalue problem. Equation (2) is Laplace's Tidal Equation; its solutions are known as Hough Functions after Hough [6] who first solved this equation in terms of series of Associated Legendre Polynomials. Its full study has, however, yet to be made. Having obtained the eigenvalues of (2), we express these in terms of equivalent depths, h_n, and use them in equation (1) which is then solved for the vertical structure of the tide. In this paper we will restrict ourselves to an isothermal atmosphere where H=constant. Then (1) becomes

$$\frac{d^2Y_n}{dx^2}\pm\lambda_n^2Y_n=\frac{\kappa J_n(x)}{\gamma gh_n}e^{-x/2} \quad (11)$$

where

$$\lambda_n^2=\left|\frac{1}{4}\left[1-\frac{4\kappa H}{h_n}\right]\right|. \quad (12)$$

The lower boundary condition for (11) is derived from the condition that

$$w_n=0 \text{ at } z=x=0. \quad (13)$$

Using (8) we see that this implies that

$$\frac{dY_n}{dx}+\left(\frac{H}{h_n}-\frac{1}{2}\right)Y_n=0 \text{ at } x=0. \quad (14)$$

The solution of (11) will have a particular part corresponding to the inhomogeneity on the right hand side, and a homogeneous solution of the form

$$A_ne^{i\lambda_nx}+B_ne^{-i\lambda_nx} \quad (15)$$

when the plus sign obtains, or

$$A_n e^{\lambda_n x} + B_n e^{-\lambda_n x} \tag{16}$$

when the minus sign obtains.

If we assume that $J_n(x) \to 0$ as $x \to \infty$, then the particular part of Y_n will also do this. Thus the particular solution will automatically satisfy any reasonable upper boundary condition. The upper boundary condition will, therefore, involve only the homogeneous solution. Two physical conditions will be invoked in order to obtain this condition. First we demand that the kinetic energy in an infinitely high column of unit cross-section be finite. As Wilkes [11] shows, this implies $A_n=0$ when expression (16) is appropriate. When (15) obtains, another condition is required. Since we have a local source of energy, a reasonable condition is that the energy flow at $x=\infty$ be strictly upward. Wilkes shows that this implies $B_n=0$ in (15).

3. SOLUTION OF LAPLACE'S TIDAL EQUATION

Recalling that we will now designate Y_n 's μ-dependence by $\Theta_n(\mu)$, we must replace Y_n by Θ_n in equation (2). Following Hough [6] we expand Θ_n in terms of Associated Legendre Polynomials P^s_m, where $s=1$, $m \geq 1$. m odd corresponds to functions symmetric about the equator. Thus

$$\Theta_n = \sum_{m=1}^{\infty} C_m^{(n)} P_m^1. \tag{17}$$

The properties of Associated Legendre Polynominals are extensively discussed in the literature (Hough [6], Morse and Feshbach [9], Belousov [1]). When (17) is substituted into (2) and s is set equal to one, one gets the following set of equations for $C_m^{(n)}$:

$$\frac{(m-1)}{(2m-1)\left(m(m-1)-\frac{f}{\omega}\right)} \left[\frac{(m-2)}{(2m-3)} C_{m-2}^{(n)} + \frac{(m-1)^2(m+1)}{m^2(2m+1)} C_m^{(n)}\right] - \left[\frac{\omega^2}{f^2}\frac{m(m+1)-\frac{f}{\omega}}{m^2(m+1)^2} - \frac{hg}{f^2a^2}\right] C_m^{(n)} + \frac{(m+2)}{(2m+3)\left\{(m+1)(m+2)-\frac{f}{\omega}\right\}} \left[\frac{(m+2)^2 m}{(m+1)^2(2m+1)} C_m^{(n)} + \frac{(n+3)}{2n+5} C_{m+2}^{(n)}\right] = 0 \tag{18}$$

where $m=1, 3, \ldots$ for symmetric functions and $m=2, 4, \ldots$ for asymmetric functions.

$$C_m^{(n)}=0, \qquad m<1.$$

Hough [6] simplified (18) in an illuminating fashion by introducing auxiliary functions D_m, defined as follows:

$$\frac{2(m+1)^2(m-1)}{m^2(2m-1)} C_{m-1}^{(n)} + \frac{2(m+2)}{(2m+3)} C_{m+1}^{(n)} = \left\{\frac{2m(m+1)}{m^2} - \frac{2}{m^2}\frac{f}{\omega}\right\} D_m^{(n)}. \tag{19}$$

Equation (19) becomes

$$\frac{2(m+1)^2(m-1)}{m^2(2m-1)} D_{m-1}^{(n)} + \frac{2(m+2)}{(2m+3)} D_{m+1}^{(n)} = \left[2\frac{\omega^2}{f^2m^2}\left\{m(m+1)-\frac{f}{\omega}\right\} - 2(m+1)^2\frac{hg}{f^2a^2}\right] C_m^{(n)}. \tag{20}$$

Equations (19) and (20) represent an infinite number of linear, homogeneous equations in an infinite number of unknowns:

$$C_1^{(n)}, C_3^{(n)}, C_5^{(n)}, \ldots \text{ and } D_2^{(n)}, D_4^{(n)}, D_6^{(n)}, \ldots$$

for the case of symmetric eigenfunctions. In order that a solution to (20) and (21) exist, the infinite determinant of their coefficients must equal zero: i.e.,

$$\begin{vmatrix} -M_1 & L_1 & 0 & 0 & 0 & \cdots \\ K_2 & -N_2 & L_2 & 0 & 0 & \cdots \\ 0 & K_3 & -M_3 & L_3 & 0 & \cdots \\ 0 & 0 & K_4 & -N_4 & L_4 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix} = 0, \tag{21}$$

where

$$K_n = 2\frac{(n+1)^2(n-1)}{n^2(2n-1)}, \tag{22}$$

$$L_n = 2\frac{(n+2)}{(2n+3)}, \tag{23}$$

$$M_n = \frac{2\omega^2}{f^2n^2}\left\{n(n+1)-\frac{f}{\omega}\right\} - 2(n+1)^2\frac{hg}{f^2a^2}, \tag{24}$$

and

$$N_n = \frac{2n(n+1)}{n^2} - \frac{2}{n^2}\frac{f}{\omega}. \tag{25}$$

In the consideration of (21), a few comments on the difference between Hough's objectives and ours are in order. Hough was dealing with the free oscillations of an ocean of constant depth. Thus, he sought those values of ω which would satisfy (21). In our problem, we are fixing ω, and seeking those values of h which satisfy (21). Hough found that the solutions of (21) were associated with the solutions of

$$M_n=0, \tag{26}$$

and

$$N_n=0. \tag{27}$$

The former, known as solutions of the first kind, correspond to oscillations wherein inertial terms dominate Coriolis terms. The latter, known as solutions of the second kind, correspond to Rossby waves where Coriolis terms are, of course, very important. In our case where ω is fixed and h is allowed to vary, M_n can still have a zero, but N_n is a constant and, hence, cannot have a zero. Siebert [10] and Haurwitz [5] among others, therefore, restrict themselves, when dealing with fixed periods, to solutions of the first kind. As may be seen from (24), these are associated with positive h's. The N_n's, however, are not unimportant. It turns out that their presence, and the presence of the nondiagonal terms in (21) give rise to additional solutions, not associated with the zeroes of M_n, and for $\omega=\Omega$ and $s=1$, involving negative h's.

The easiest way to see this is to outline the actual solution of (21). In order for (17) to converge, C_m must approach zero as $m\rightarrow\infty$. According to (20) so must D_m. Therefore, a given solution of (21) is also an approximate solution of equation (21) when the determinant is truncated after a sufficiently large number of terms. Let $x=hg/f^2a^2$ and let $D_l(x)=$the $l\times l$ truncation of the determinant in equation (21). In practice we solve $D_l(x)=0$, and see if the solution is considerably altered by considering $D_{l+1}(x)=0$. We continue this process until the root is negligibly altered. We then substitute the value of x obtained into equations (19) and (20) and solve for C_m. For example, consider

$$D_2(x)=0, \tag{28}$$

with $s=1$ and $\omega=\Omega=\frac{1}{2}f$; $M_1=-8x$, $L_1=1.2$, $K_2=1.5$, and $N_2=2$. The solution of (28), therefore, is

$$x=-0.1125.$$

We now continue the process and find that $D_7(x)=0$ yields $x=-0.139$, and $D_{12}(x)=0$ yields $x=-0.1390$. In this manner we have found the first two negative solutions of (21), and their associated eigenfunctions. The two solutions are designated by the subscripts -1 and -3. Haurwitz [5] has found the first three positive solutions and these will be designated by the subscripts $+3$, $+5$, and $+7$. These values of x together with their associated equivalent depths are given in table 1; also shown are the λ_n's (see equation (12)).

The idea of a negative equivalent depth may seem unreasonable, but, in fact, the name—equivalent depth—is misleading. What meaning it has may be seen from a study of equations (11), (12), (15), and (16) in conjunction with equations (6), (7), (8), and (10). Assume that the plus sign obtains in (11) and that the homogeneous solution (15) is therefore appropriate. We see from equations (6), (7), (8), and (10) that the amplitudes of the u, v, w, and δT fields increase with altitude as $e^{z/2}$. When, however, h is negative, the negative sign obtains in (11), and moreover, $\lambda>\frac{1}{2}$. Therefore, in this case the amplitudes of the u, v, w, and δT fields actually decrease with altitude.

TABLE 1.—*Eigenvalues of Laplace's Tidal Equation and associated numbers ($\omega=\Omega$, $s=1$)*

n	-1	-3	$+3$	$+5$	$+7$
x_n	-0.1390	-0.1987×10^{-1}	$+0.7930\times10^{-2}$	$+0.1380\times10^{-2}$	$+0.0555\times10^{-2}$
h_n	-0.1225×10^{5}	-0.1751×10^{4}	$+0.6987\times10^{3}$	$+0.1216\times10^{3}$	$+0.4890\times10^{2}$
λ_n	0.6643	1.2601	1.762	4.361	6.904

We will express the Hough functions corresponding to the above-described equivalent depths as follows:

$$\Theta_n=\sum_{m=1} R^n_m \overline{P}^1_m(\mu), \tag{30}$$

where P^1_m is a normalized Associated Legendre Polynomial (tabulated in Belousov [1])

$$\int_{-1}^{1} [\overline{P}^m_n(\mu)]^2 d\mu=1, \tag{31}$$

and

$$\overline{P}^m_n(\mu)=\sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}\,P^m_n(\mu). \tag{32}$$

We have also set

$$\sum_{m=1} (R^n_m)^2=1, \tag{33}$$

thus normalizing the Θ_n's. The normalization will, of course, be slightly changed when we include more terms in (30). The R^n_m's are given in table 2. The Θ_n's themselves are shown in figure 1. For positive n, Θ_n has increasingly less amplitude outside the critical latitudes as n increases, while for negative n, Θ_n has increasingly less amplitude within the critical latitudes as n increases. However, Θ_{-1} and Θ_{-3} still have considerable amplitude within the critical latitudes. Therefore, if we wish to expand a function with amplitude distributed throughout the latitude domain in terms of Θ_{-1}, Θ_{-3}, Θ_{+3}, Θ_{+5}, and Θ_{+7}, we should get a good representation near the equator and a poorer representation near the poles.

TABLE 2.—*Coefficients R^n_m for expansion of Θ in terms of $\overline{P}^1_m$*

m	n				
	-1	-3	$+3$	$+5$	$+7$
1	0.8968	-0.2703	0.2842	-0.0738	0.0378
3	0.4401	0.4613	-0.6402	0.1573	-0.0800
5	0.04534	0.7731	0.6195	-0.0594	0.0239
7	0.0020	0.3302	-0.3337	-0.2401	0.1331
9	.	0.0698	0.1152	$+0.5150$	-0.2388
11	.	0.0081	-0.0276	-0.5807	0.1500
13	.	.	0.0049	$+0.4574$	0.1089
15		.	-0.0007	-0.2757	-0.3795
17		.	0.0001	$+0.1332$	0.5197
19			.	-0.0530	-0.4977
21			.	0.0177	0.3785
23			.	-0.0051	-0.2386
25				0.0012	0.1283
27				.	-0.0591
29				.	0.0216

4. EXPANSION OF OBSERVED PRESSURE

Since the latitude dependence of observed tidal fields is almost always expressed in terms of Associated Legendre Polynomials, it proves useful to expand these in terms of Hough Functions. This is rendered particularly simple as a result of the use of normalized Hough Functions. It is easily shown that

$$\bar{P}^1_m = \sum_n R^n_m \Theta_n(\mu). \tag{34}$$

In figure 2 we show $\bar{P}^1_1$, P^1_3 and their representations in terms of Θ_{-1}, Θ_{-3}, Θ_{+3}, Θ_{+5}, and Θ_{+7}. As suggested in section 3, the representations are quite accurate near the equator, but relatively poor at higher latitudes. Nevertheless, the nature of the approximation and its convergence are clearly indicated in the figure, and we may safely use the approximate representations.

As an example, let us consider Haurwitz's [5] representation of the solar diurnal oscillation in the surface pressure (in μb.):

$$S^1_1(P_0) = 535.8[\bar{P}^1_1 - 0.2966\bar{P}^1_3] \sin(t+\varphi+12°). \tag{35}$$

With the use of (34), (35) may be approximately represented as follows:

$$S^1_1(P_0) = \{410.6\Theta_{-1} - 218.1\Theta_{-3} \ldots + 254.0\Theta_{+3} - 64.5\Theta_{+5} + 32.97\Theta_{+7} \ldots\} \sin(t+\varphi+12°). \tag{36}$$

In figure 3, we show $S^1_1(P_0)$ as given by (35), as approximated by (36), and Haurwitz's representation in terms of the first three Hough Functions of the first kind. Clearly the last is an extremely poor representation. Equation (36) gives a much better representation—oscillating about the actual curve. The accuracy of the representation near the equator suggests that similar accuracy could be obtained at high latitudes by the inclusion of another two Hough Functions of the second kind. The present representation is, however, adequate for our purposes.

5. SURFACE PRESSURE OSCILLATION DUE TO SOLAR INSOLATION

According to Siebert [10], the diurnal heating due to insolation is given by an expression of the form

$$J = \frac{i\omega R}{\kappa M} e^{-z/3} \sum_m \hat{\tau}_m \bar{P}^1_m(\mu) \sin(t+\varphi). \tag{37}$$

It is a simple matter, using (34), to rewrite (37) as

$$J = \frac{i\omega R}{\kappa M} e^{-z/3} \sin(t+\varphi) \sum_m \tau_m \Theta_m(\mu). \tag{38}$$

Then, the J_n's in (11) become

$$J_n = \frac{i\omega R}{\kappa M} \tau_n e^{-z/3}, \tag{39}$$

and (11) becomes

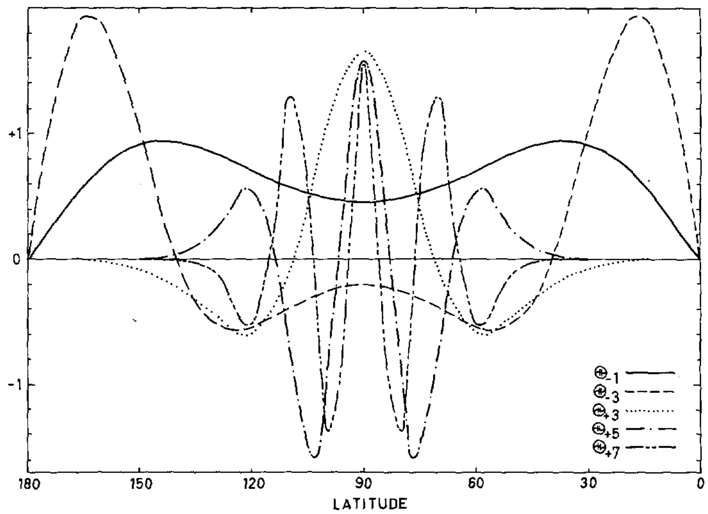

FIGURE 1.—Θ_{-1}, Θ_{-3}, Θ_{+3}, Θ_{+5}, and Θ_{+7} as functions of latitude.

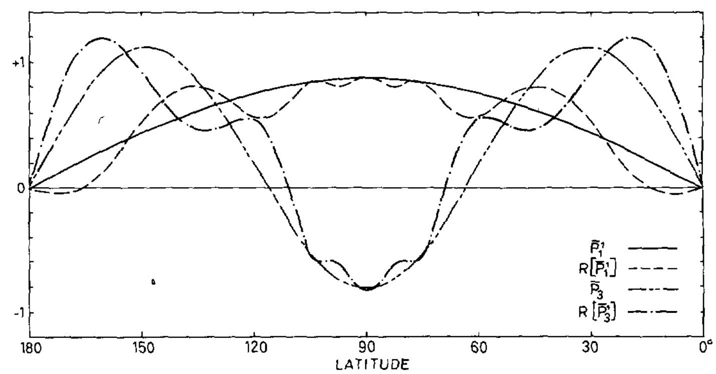

FIGURE 2.—$\bar{P}^1_1$, $\bar{P}^1_3$, and their representations in terms of Θ_{-1}, Θ_{-3}, Θ_{+3}, Θ_{+5}, and Θ_{+7}.

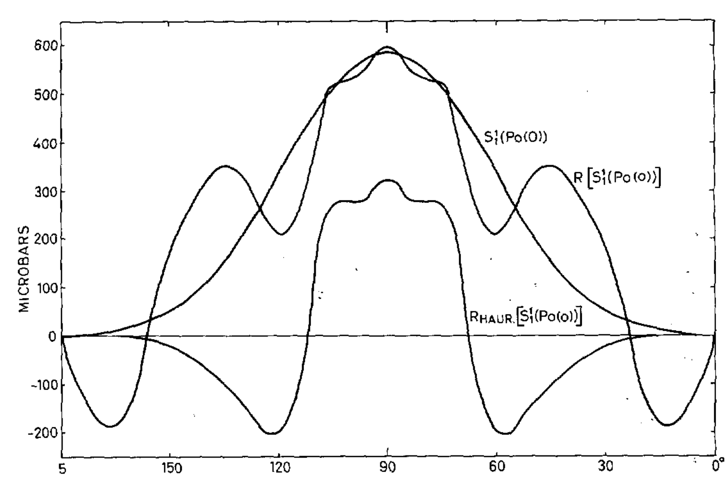

FIGURE 3.—$S^1_1(p_0)$, its representation in terms of five Hough Functions, and its representation by Haurwitz in terms of Θ_{+3}, Θ_{+5}, and Θ_{+7}.

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$$\frac{d^2Y_n}{dx^2} \pm \lambda_n^2 Yn = \frac{i\omega R}{\gamma g h_n M} \tau_n e^{-5x/6}. \tag{40}$$

For positive h_n, the plus sign is appropriate and, using the boundary conditions indicated in section 2, one gets

$$Y_n = \frac{\frac{i\omega}{\gamma}\frac{H}{h_n}\frac{\tau_n}{T}}{\left(\frac{25}{36}+\lambda_n^2\right)} \left\{ \frac{\frac{4}{3}-\frac{H}{h_n}}{i\lambda_n + \frac{H}{h_n} - \frac{1}{2}} e^{i\lambda_n x} + e^{-5x/6} \right\}. \tag{41}$$

For negative h_n, (40)'s solution is

$$Y_n = \frac{\frac{i\omega}{\gamma}\frac{H}{h_n}\frac{\tau_n}{T}}{\left(\frac{25}{36}-\lambda_n^2\right)} \left\{ \frac{\frac{4}{3}-\frac{H}{h_n}}{\frac{H}{h_n} - \frac{1}{2} - \lambda_n} e^{-\lambda_n x} + e^{-5x/6} \right\}. \tag{42}$$

We wish to obtain the surface pressure oscillation resulting from the J_n's. It is easy to obtain from equations (9) and (14) the well known result

$$\delta p_n(0) = \frac{i\gamma}{\omega} P_0(0) Y_n(0), \tag{43}$$

which yields for positive h_n

$$\delta p_n(0) = -\frac{P_0(0)\frac{H}{h_n}\frac{\tau_n}{T_0}}{\left(\frac{5}{6}-i\lambda_n\right)\left(i\lambda_n + \frac{H}{h_n} - \frac{1}{2}\right)}, \tag{44}$$

and for negative h_n

$$\delta p_n(0) = -\frac{P_0(0)\frac{H}{h_n}\frac{\tau_n}{T_0}}{\left(\frac{5}{6}+\lambda_n\right)\left(\frac{H}{h_n} - \frac{1}{2} - \lambda_n\right)}. \tag{45}$$

Equations (44) and (45) may be rewritten

$$\delta p_n(0) = \zeta_n \tau_n. \tag{46}$$

We will take $T_0 = 280°$ and $P_0(0) = 1000$ mb. H becomes 8.2 km. Using these values and the results in table 1, we may calculate the ζ_n's. These are given in table 3.

There remains the problem of selecting the τ_n's. Several sets are available from the literature, though not all are appropriate. In table 4 we list sets of τ_n's deduced from the semiempirical estimates of Haurwitz [4], and the theoretical estimates of Kertz [7] and Siebert [10]. Siebert's values are due solely to insolation, while Kertz's values include both radiation and turbulent transfer. Similarly, Haurwitz's values include these effects. A simple correction may be applied to Haurwitz's values. We may subtract Siebert's values from those of Kertz in order to obtain an estimate of the effect of processes other than insolation. These quantities may then be subtracted from Haurwitz's values in order to obtain semiempirical esti-

TABLE 3.—*Coefficients ζ_n relating δp_n to τ_n*

n	−1	−3	+3	+5	+7
Amplitude ζ_n (μb. deg.$^{-1}$)	8.707×10^2	1.240×10^3	1.891×10^3	8.113×10^2	5.141×10^2
Phase ζ_n (deg.)	180	180	236	255.3	260.5

TABLE 4.—*Various τ_n's*

n	Haurwitz		Kertz		Siebert		Haurwitz (corrected)		Needed to yield observed $\delta p(0)$	
	Amp. (°K.)	Ph. (deg.)	Amp. (°K.)	Ph. (deg.)	Amp. (°K.)	Ph. (deg.)	Amp. (°K.)	Ph. (deg.)	Amp. (°K.)	Ph. (deg.)
−1	1.211	233	0.803	225	0.157	180	0.502	233	0.472	192
−3	0.141	45	0.204	45	0.055	0	0.100	259	0.176	12
+3	0.098	216	0.207	225	0.062	180	0.167	73	0.134	136
+5	0.029	38	0.054	45	0.016	0	0.040	254	0.079	297
+7	0.015	218	0.028	225	0.008	180	0.020	74	0.064	111

mates for the insolation effects. The resulting values are included in table 4 too. Also included in table 4 are those values of τ_n that would result in the observed pressure as given by equation (36).

We see from table 4 that Siebert's values for the τ_n's are too small by approximately a factor of three to yield the observed diurnal surface pressure oscillation. Although Haurwitz's [5] average observed pressure distribution is only a relatively crude fit to widely scattered data, errors on this order seem unlikely. On the other hand, Haurwitz's [4] corrected values of τ_n appear to be close to the necessary amplitudes. There are, however, considerable errors in their phase. In some recent calculations which the author will publish shortly, it is shown that the contribution of ozone heating to the surface pressure oscillation is by no means insignificant (ca. 170 μb.), and hence these discrepancies should not be surprising.

6. CONCLUSIONS

It is found that the Hough Functions computed by Haurwitz are not complete and hence the fact that they are not suitable for representing the observed data is not surprising. When the additional Hough Functions corresponding to negative equivalent depths are added to the set, the completeness of the set and its suitability for representing the observed data become evident.

As an application of the above results, we have investigated the diurnal surface pressure oscillation that should result from solar insolation. In agreement with current notions in tidal theory we find that there is nothing surprising about the small amplitude of the observed surface pressure oscillation. Quite the reverse—we find that Siebert's insolation model can account for only about a third of the observed diurnal pressure oscillation. However, a crudely modified version of Haurwitz's [4] semiempirical model for heating results in approximately the observed amplitude for δp; insolation remains the likely main cause for the surface pressure oscillation.

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